MAPPING CYLINDER NEIGHBORHOODS(1)

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1. Let X be a triangulated 3-manifold and C a subcomplex of X. A regular neighborhood of C in X is the union of all simplexes in a second derived subdivision of X that intersect C. Every subcomplex C of X has a regular neighborhood. We consider the converse using a generalization of regular neighborhoods.

Let C be a closed subset of a space X. A subspace U of X is called a *mapping cylinder neighborhood* (MCN) of C if $U=f(M\times I)\cup C$ where f is a map of a space $M\times I$ into X such that $f|M\times[0,1)$ is a homeomorphism into $X-C$, $f(M\times 1)=C\cap \mathrm{Cl}\,(X-C)$ and $f(M\times(0,1])\cup C$ is open in X. As noted in [12], regular neighborhoods are MCN's.

Suppose C is a closed subset of a 3-manifold X and $U=f(M\times I)\cup C$ a MCN of C. We note some properties of U.

(a) Since $M\times(0,1)$ is a 3-manifold, M is a generalized 2-manifold [17] and thus a 2-manifold [19]. Hence U is a 3-manifold with boundary.

(b) If C is compact then U is compact (Lemma 1).

(c) If C is compact and U' is another MCN of C then Int U and Int U' are homeomorphic [12]. Thus U and U' are homeomorphic [9, Theorem 3].

Our converse: Suppose X is a 3-manifold and $C\subset X$ is a topological complex, i.e., C is homeomorphic to a locally finite simplicial complex. Suppose also that C is closed in X and C has a MCN. Then C must be a subcomplex of some triangulation of X.

THEOREM 1. *If C is a topological complex which is a closed subset of a 3-manifold X, then C is tame if and only if C has a MCN.*

Our motivation for Theorem 1 was the special case where C is a 1, 2 or 3-cell and M is a 2-sphere [6], [10]. An immediate corollary to Theorem 1 is

THEOREM 2. *Suppose C is a tame topological complex in a 3-manifold X, g is a map of X into a 3-manifold Y such that $g^{-1}g(C)=C$, g is a homeomorphism on $X-C$, and $g(C)$ is a topological complex. Then $g(C)$ is tamely embedded in Y.*

Proof. By Theorem 1, C has a MCN U. The conditions on g guarantee that $g(U)$ is a MCN of $g(C)$.

A special case of Theorem 1 in dimension four is also immediate. Suppose N is a space, $f\colon N\to N$ an onto map, and N_f the mapping cylinder defined by N and

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f. Let $g: N\times I \to N_f$ be the natural map. If there exists a pseudo-isotopy $k_t(N) \to N$ such that $k_0=\text{id}$ and $k_1=f$, then the map $h: N_f \to N\times I$ defined by $h[g(x, t)] =(k_t(x), t)$ is a homeomorphism. If G is a cellular upper semicontinuous decomposition of a 3-manifold M and M/G is a 3-manifold, then there exists a pseudo-isotopy of M onto itself that shrinks the nondegenerate elements to points [18]. Thus such a pseudo-isotopy exists when N and $f(N)$ are 3-manifolds and f is the projection map of a cellular upper semicontinuous decomposition of N. We have

THEOREM 3. *Suppose N is a compact connected 3-manifold in a 4-manifold Y. Suppose M is a 3-manifold and $U=f(M\times I)$ is a MCN of N where the restriction of f to each component of $M\times 1$ is a cellular map. Then U is a bicollar for N in Y.*

Is the cellularity condition given in Theorem 3 implied by the fact that Y is a 4-manifold? This is the case in dim 3; see Lemma 3(2).

2. **Proof of Theorem 1.** We have placed the lemmas in the sections following the proof.

Proof. Suppose C has a MCN. By the procedure described at the first of the proof of Lemma 6 the union of the 1 and 2-skeleton of C has a MCN. Thus the interior of each 2-simplex is tame by Lemma 3(3). By Lemmas 5 and 6 the 1-skeleton of C is tame. In particular, the boundary of each 2-simplex in C is tame. It is a consequence of Lemmas 5.1 and 5.2 of [13] that a disk is tame if its interior and boundary are tame. Therefore the star of each vertex in C is tame, since each 2-simplex is tame and the 1-skeleton is tame [7, Theorem 3.3]. Thus C is locally tame and hence tame [3]. Suppose C is tame. Then C has a regular neighborhood under some triangulation of X. This regular neighborhood is a MCN. This completes the proof.

2.a. Suppose C is a closed subset of a 3-manifold X and $U=f(M\times I)\cup C$ is a MCN of C. We let $F=f|M\times 1$ and $i_t(t\in I)$ denote the identification of $M\times t$ with M. For example, if $x\in C\cap \text{Cl}\,(X-C)$ then $f(i_1F^{-1}(x)\times 0)\subset \text{Bd}\ U$.

LEMMA 1. *Suppose C is a closed subset of a 3-manifold X and $U=f(M\times I)\cup C$ is a MCN of C.* (1) *If A is open in $F(M\times 1)$ and contractible then every simple closed curve in $F^{-1}(A)$ separates $F^{-1}(A)$.* (2) *If $A\subset C$ is compact then $F^{-1}(A)$ is compact.*

Proof. Suppose not. There exist simple closed curves S_1 and S_2 in $f(i_1F^{-1}(A)\times 0) \subset \text{Bd}\ U$ which intersect in one point and cross there. Both curves are inessential in U because of the mapping cylinder structure over A. A simple closed curve which bounds a singular disk has a neighborhood homeomorphic to a solid torus. Thus $S_1\cup S_2$ has an orientable neighborhood in Bd U. This is a contradiction. A 3-manifold with boundary having orientable boundary cannot contain two inessential simple closed curves in its boundary which cross at an odd number of points. See [11, p. 29] or [15, Lemma 6.1].

(2) The local compactness of the MCN implies that F is a compact map. For let T be any compact subset of C. There exists an open set $Q\subset U$ such that $T\subset Q$

$\subset \bar{Q} \subset \text{Int } U$ and $\bar{Q}$ is compact. For each point $p \in F^{-1}(T)$, the arc $f(i_1(p) \times I)$ intersects $\bar{Q} - Q$. Since $F^{-1}(T)$ is closed, $J = f(i_1 F^{-1}(T) \times I) \cap (\bar{Q} - Q)$ is compact. Thus $F^{-1}(T)$ is compact, since the projection of $M \times I$ onto $M \times 1$ carries $f^{-1}(J)$ onto $F^{-1}(T)$.

2.b. *2-simplexes.* First some definitions. Let $x \in L$ where L is a 2-manifold with boundary in a 3-manifold X. The *local separation theorem* [1, §2, Corollary 2] yields: For every $\varepsilon > 0$, there exists an ε-neighborhood N of x in X such that $N - L$ has two components O_1 and O_2. If $x \in \text{Bd } L$, then $O_2 = \varnothing$. If $x \in \text{Int } L$, then O_1 and O_2 are nonempty. We say $U' \subset X$ is a 1*-sided neighborhood* of x if there exists a neighborhood N of x from the local separation theorem such that $O_1 \cup (N \cap L) \subset U'$ and $O_2 \cap U' = \varnothing$.

Let C be a topological complex which is closed in X and consists of 1 and 2-simplexes. Let $U = f(M \times I)$ be a MCN of C and Δ a 2-simplex of C. We say U contains a 1*-sided MCN*, U', of $x \in \text{Int } \Delta$ if there exists a disk $D \subset M$ such that $U' = f(D \times I)$ is a 1-sided neighborhood of X. We shall show in Lemma 3 that 1-sided MCN's always exist. Lemma 2 is a standard type of result for 2-manifolds; we omit a proof.

LEMMA 2. *Let M be a* 2*-manifold, B a nonempty, proper open connected subset of M such that $\bar{B}$ is compact and every simple closed curve in B separates B. If E is a continuum in B and K is a continuum in B which separates E from* Bd B, *then E lies in the interior of a disk $D \subset B$.*

Consider a fixed $x \in \text{Int } \Delta$. We distinguish two sets, H and L, in $M \times 1$ which correspond to the two sides of Δ near x. Let N, O_1 and O_2 be given for x by the local separation theorem. Let P be a disk such that $x \in \text{Int } P \subset P \subset (N \cap \text{Int } \Delta)$ and $z \in \text{Int } P$. For $y \in F^{-1}(z)$, let A_y denote the arc $f(i_1(y) \times I)$. There exists a first point p from z in $A_y \cap (\bar{N} - N)$. Then $[z, p) \subset N$ and $(z, p) \subset O_1$ or O_2. We say A_y *ends through* O_1 or O_2, respectively. Let $H_z(L_z)$ be the set of all points y such that A_y ends through $O_1(O_2)$. Let $H = \bigcup H_z$, $L = \bigcup L_z$, $z \in \text{Int } P$.

We show H and L are open and separated. We assume the neighborhood N was chosen to lie inside a neighborhood Q of x homeomorphic to E^3. Suppose there exist $y \in H$ and $b \in L$ lying in the same component of $F^{-1}(\text{Int } P)$. There exist an arc $by \subset F^{-1}(\text{Int } P)$ and an arc $F(b)F(y) \subset \text{Int } P$. There exists $0 < t < 1$ so that the arc $f(i_1(by) \times t)$ together with $F(b)F(y)$ and subarcs of A_y and A_b form a simple closed curve $S \subset Q$. Since A_y ends through O_1 and A_b ends through O_2, S links Bd P (homology linking mod 2; see [4]). But since $by \subset F^{-1}(\text{Int } P)$, S can be shrunk to a point in $Q - \text{Bd } P$ by first pulling it into Int P using the mapping cylinder. Contradiction. Therefore H and L are the union of components of $F^{-1}(\text{Int } P)$. Thus they are open and separated.

LEMMA 3. *Suppose C is a topological complex which is a closed subset of a* 3*-manifold X. Suppose C consists of* 1 *and* 2*-simplexes and $U = f(M \times I)$ is a MCN of C. Also suppose Δ is a* 2*-simplex in C and $x \in$* Int Δ. *Then* (1) *U contains a* 1*-sided MCN* *of x on each side of Δ and the two disks defining the MCN's are disjoint*, (2) H_x *and* L_x *are cellular in* M, *and* (3) Int Δ *is locally tame.*

Proof. (1) Let x be a distinguished point in Int Δ and N, O_1 and O_2 be given for x by the local separation theorem such that N lies in a neighborhood of x homeomorphic to E^3. Let P be a disk such that $x \in \text{Int } P \subset P \subset (N \cap \text{Int } \Delta)$ and let H and L be given as in the discussion preceding the lemma. We consider only H. By Lemma 1, $F^{-1}(x)$, and hence H_x, is compact. If H_x were not connected we could separate two of its components, say T_1 and T_2, in $M \times 1$ with a finite number of simple closed curves $S_i \subset H$. But points in $f(T_1 \times I)$ and $f(T_2 \times I)$ can be joined by small arcs in O_1. A contradiction is reached since $f(\bigcup S_i \times [0, 1))$ separates $f(M \times [0, 1))$ and x is not a limit point of $f(\bigcup S_i \times I)$. Thus $F|H$ is monotone. There exist disks D_1 and D_2 such that $x \in \text{Int } D_1 \subset D_1 \subset \text{Int } D_2 \subset D_2 \subset \text{Int } P$. Let $E = H_x$ and $B = \bigcup H_y$, $y \in \text{Int } D_2$. Let $K = \bigcup H_y$, $y \in \text{Bd } D_1$. The map F is closed on $F^{-1}(D_2) \cap H$. The inverse image of a connected set is connected under a monotone closed map. Thus the sets E, K and B satisfy the hypothesis of Lemma 2. Let $D \subset M \times 1$ be a disk given by Lemma 2. Then $U' = f(i_1(D) \times I)$ is a 1-sided MCN of x. For since $D \subset H$, there exists $t < 1$ such that $f(i_1(D) \times [t, 1]) \subset O_1$. Picking a neighborhood $N(q)$ of x by the local separation theorem such that $N(q) \subset N$ and $N(q) \cap f(i_1(D) \times [0, t]) = \varnothing$, we have $O_2(q) \cap f(i_1(D) \times I) = \varnothing$. Since $H_x \cap \text{Bd } D = \varnothing$ and $i_1(\text{Bd } D) \times I$ separates $M \times I$, there exists a neighborhood $N(r)$ of x from the local separation theorem such that $O_1(r) \subset U'$. For a neighborhood $N(s)$ of x from the local separation theorem contained in $N(q) \cap N(r)$ we have $O_1(s) \subset U'$ and $O_2(s) \cap U' = \varnothing$. Therefore U' is a 1-sided MCN of x. A similar argument using L yields a disk disjoint from D and a 1-sided MCN of x on the O_2 side of Δ.

(2) Let $N(s)$ be the neighborhood of x given above and D_3 a disk such that $x \in \text{Int } D_3 \subset D_3 \subset (N(s) \cap \text{Int } \Delta)$. Then $H \cap F^{-1}(\text{Int } D_3)$ is an open connected subset of Int D and not separated by H_x. Thus H_x, and similarly L_x, is cellular.

(3) We shall show that $X - \text{Int } \Delta$ is locally simply connected at x. Consider the 1-sided MCN of x, $U' = f(i_1(D) \times I)$ and let $\varepsilon > 0$. There exists an ε-neighborhood $N(\varepsilon)$ of x from the local separation theorem such that $O_1(\varepsilon) \subset U'$ and $O_2(\varepsilon) \cap U' = \varnothing$. Since $F^{-1}(N(\varepsilon) \cap U')$ is open in $i_1(D) \times I$ and H_x is cellular, there exists a disk $G \subset i_1(D)$ and a number $t < 1$ such that $G \times [t, 1] \subset F^{-1}(N(\varepsilon) \cap U')$ and $H_x \subset \text{Int } G \times 1$. Let $T = (\text{Int } G) \times (t, 1]$. There exists a neighborhood Q of x from the local separation theorem such that $Q \subset N(\varepsilon)$ and $Q \cap f((D \times I) - T) = \varnothing$. The component $O_1(q)$ of $Q - \text{Int } \Delta$ lies in $f(T)$. Let J be any simple closed curve in $O_1(q)$. There exists $r < 1$ such that $f(D \times r)$ separates J from $F(D)$ in $f(i_1(D) \times I)$. Thus J can be shrunk to a point in the interior of the 3-cell $f(G \times [t, r]) \subset N(\varepsilon)$. Using the 1-sided MCN of x on the other side of Int Δ, we have that $X - \text{Int } \Delta$ is locally simply connected at x. Since $X - \text{Int } \Delta$ is locally simply connected at each $x \in \text{Int } \Delta$, Int Δ is locally tame [5].

2.c. 1-*complexes*. Let n be a positive integer. An n-frame T is the union of n-arcs $A_i=[p, a_i]$ such that $A_i \cap A_j=p$. The points a_i are the endpoints of T. The interior of T, Int T, is T minus its endpoints. We define a *MCN of the interior of* T. No confusion should result from this different use of MCN. Let S^2 denote the 2-sphere and D_i, $i=1, \dots, n$, be disjoint disks in S^2. Let $M=S^2-\bigcup D_i$ and consider $M \times I$ as a subspace of $S^2 \times I$. If T is an n-frame in a 3-manifold X then Int T is said to have a MCN, $U=f(M \times I)$, if there exists a map f of $M \times I$ into X such that (1) $f|M \times [0, 1)$ is a homeomorphism into $X-T$, (2) $f(M \times 1)=\text{Int } T$, (3) U is a neighborhood of Int T in X, and (4) for any sequence $\{b_j\}$ in $M \times I$ which converges to a point of Bd $D_i \times 1$, $\{f(b_j)\}$ converges to the endpoint a_i of T.

LEMMA 4. *Suppose T is an n-frame in a* 3-*manifold X. If there exists a MCN, $f(M \times I)$, of* Int T *then* Int T *is locally tame.*

Proof. The proof of Lemma 4 follows the procedure used to prove Theorem 1 in [6]. We partition a neighborhood of Int T and a neighborhood of the interior of a standard n-frame in E^3 into homeomorphic pieces. We then obtain a homeomorphism between the neighborhoods which carries T onto the standard n-frame. Since for each $t \in (0, 1)$, $f(M \times t)$ is bicollared, we may assume that $f(M \times 0)$ is locally tame. Let C be a circle, $A=C \times (0, 1) \times I$ and $(x, y, z) \in A$ such that $x \in C$, $y \in (0, 1)$ and $z \in I$. Let B denote the half-open annulus in A, $B=\{(x, y, z) : y =1/2z+1/2\}$.

The properties given in Lemma 1 also hold for a MCN of Int T. It therefore follows that F is closed and monotone. Thus the inverse image under F of any connected subset of Int T is connected. For each i, $F^{-1}(\text{Int } A_i)$ is a component of $(M \times 1)-F^{-1}(p)$ and $F^{-1}(\text{Int } A_i) \cup D_i$ is a component of $(S^2 \times 1)-F^{-1}(p)$. Since each component of S^2 minus the continuum $F^{-1}(p)$ is homeomorphic to E^2, subtracting the disk D_i yields that $F^{-1}(\text{Int } A_i)$ is an open annulus. Thus there exist homeomorphisms k_1 and k_2 of A into $i_1F^{-1}(\text{Int } A_i) \times I$ such that $B(a_i)=\text{Cl }(fk_1(B))$ is a disk with $B(a_i) \cap T=a_i$, $B(p_i)=\text{Cl }(fk_2(B))$ is a disk with $B(p_i) \cap T=p$, Int $B(a_i)-a_i$ and Int $B(p_i)-p$ are locally tame, and $B(p_i) \cap B(a_i)=\varnothing$. Similarly, for each $x \in \text{Int } A_i$, $F^{-1}(\text{Int } A_i)-F^{-1}(x)$ is the union of two disjoint open annuli. By mapping A homeomorphically into one of these annuli we can define a disk $B(x)$ such that $B(x) \cap T=x$ and $B(x)-x$ is locally tame. For distinct x and y in A_i there exist numbers $t_1<t_2<1$ such that if $W=\text{Cl }(f(M \times [t_1, 1])-f(M \times [t_2, 1]))$ and if P is the closure of the component of $f(M \times I)-(B(x) \cup B(y))$ that intersects Int A_i then $W \cap P$ is a tame solid torus. Let O be the closure of the component of $f(M \times I)-\bigcup B(a_i)$ that contains Int T. Let L_i be the closure of the component of $O-\bigcup B(p_i)$ that contains Int A_i and $L=\text{Cl }(O-\bigcup L_i)$. Let O' be the unit ball in E^3 and T' an n-frame whose vertex is the origin, whose endpoints lie in Bd O' and which is composed of straight line segments. Partition O' into regions L' and L'_i corresponding to L and L_i. It follows from the proof of Theorem 1 of [6] that we may partition L_i-A_i and $L'_i-A'_i$ into tame solid tori as above whose diameters go to zero as the tori approach T in such a way that a homeomorphism $R_i: L_i \to L'_i$ can be obtained by defining homeomorphisms on corresponding tori. Let $J_1 = f(M \times [0, 1/2]) \cap L$, $J_j = f(M \times [1/j, 1/j+1]) \cap L$, $j \geqq 2$. Each J_j $(j \geqq 1)$ has tame boundary and is homeomorphic to $M \times I$. There exists a collection of regions $\{J'_j\}$ in L' and a sequence of onto homeomorphisms S_j $(j \geqq 1)$ such that $S_j: J_j \to J'_j$ and S_j extends R_i $(i=1, \ldots, n)$ and S_k $(k<j)$. The union of the S_j and the R_i can be extended to a homeomorphism of O onto O' that carries T onto T'. Thus Int T is locally tame.

LEMMA 5. *Suppose C is a topological 1-complex which is a closed subset of a 3-manifold X. Suppose C has a MCN, $f(M \times I)$. Then C is tame.*

Proof. Let p be a vertex of C and T the n-frame consisting of all simplexes in C containing p. We shall show that $f(i_1 F^{-1}(\text{Int } T) \times I)$ is a MCN of Int T. It follows from Lemma 1 that F is closed and monotone. Let $[p, a_i]$ be a 1-simplex in T. Let $x, y \in (p, a_i)$ and $q \in (x, y)$. Let $K_z = f(i_1 F^{-1}(z) \times 0)$, $z = x$, y and q. There exist simple closed curves S and S' which separate K_q from K_x and K_q from K_y in $f(i_1 F^{-1}[x, y] \times 0)$, respectively. The curves S and S' can be shrunk to points on disjoint subsets of $f(M \times (0, 1]) \cup S \cup S'$. They therefore bound disjoint disks there by Dehn's Lemma [16]. Let K be the union of the two disks and the component of $f(i_1 F^{-1}[x, y] \times 0) - (S \cup S')$ that contains K_q. Since a simple closed curve in K can be pushed off of the two disks, we can obtain from Lemma 1 that every simple closed curve in K separates K. Thus K is a 2-sphere [2]. It follows that $F^{-1}(p, a_i)$ is an open annulus. Thus for each i, there exists a disk B_i in $f(i_1 F^{-1}(p, a_i) \times I) \cup a_i$ constructed as in the proof of Lemma 4 such that $B_i \cap T = a_i$. The component of $f(M \times 0) - \bigcup \text{Bd } B_i$ which contains $f(i_1 F^{-1}(p) \times 0)$ is a sphere with n-holes since its union with the disks B_i is a 2-sphere (again by Lemma 1 and [2]). By the construction of the B_i, $F^{-1}(\text{Int } T)$ is a sphere with n-holes. It follows that $f(i_1 F^{-1}(\text{Int })T \times I)$ is a MCN of Int T. By Lemma 4, C is locally tame and hence tame [3].

LEMMA 6. *Suppose C is a topological complex which is a closed subset of a 3-manifold X. If C has a MCN then the 1-skeleton of C has a MCN.*

Proof. Let M' be a 2-manifold and f' a map of $M' \times I$ into X such that $f'(M' \times I) \cup C$ is a MCN of C. Let $\{K_i\}$ be the collection of all 3-simplexes in C. Let N_i be a layer in the collar for Bd K_i in K_i. Let $M = M' \cup \{N_i\}$ and $f: M \times I \to X$ be such that $f|M' \times I = f'$, $f|N_i \times I$ is a homeomorphism onto the region between N_i and Bd K_i, and $f(N_i \times 0) = N_i$. Then $f(M \times I) \cup C$ is a MCN of C_0, the union of the 1- and 2-skeleton of C. Having removed the 3-simplexes we proceed to eliminate the 2-simplexes. Let Δ be a 2-simplex in C_0. We shall show there exists a 2-manifold M_1 and a map H_1 of $M_1 \times I$ into X such that $H_1(M_1 \times I)$ is a MCN of $C_0 - \text{Int } \Delta$ and $H_1(M_1 \times I)$ agrees with $f(M \times I)$ outside of $f(i_1 F^{-1}(\text{Int } \Delta) \times I)$. Defining such a map for each 2-simplex in C_0 will yield a MCN of the 1-skeleton of C. Let

$p \in \text{Int } \Delta$. By Lemma 3 there exist disjoint disks D_i $(i=1, 2)$ in M such that $F(D_i \times 1) \subset \text{Int } \Delta$ and $U_i = f(D_i \times I)$ are 1-sided MCN's of p on opposite sides of Δ. Let Δ_1 and Δ_2 be disks lying in the intersection of the interiors of $F(D_1 \times 1)$ and $F(D_2 \times 1)$ such that $p \in \text{Int } \Delta_1 \subset \Delta_1 \subset \text{Int } \Delta_2$.

The MCN of $C_0 - \text{Int } \Delta_2$. Intuitively, we bore a hole through the MCN. Let $A = F^{-1}(\text{Bd } \Delta_1) \cap (D_1 \times 1)$ and $J = F^{-1}(\text{Bd } \Delta_2) \cap (D_1 \times 1)$. The region between A and J is an open annulus. Let B be a simple closed curve lying in this region and concentric to A and J. There exists a map k of $D_1 \times I$ onto itself that carries each region in Figure 1 onto the corresponding region (labeled with a prime) in Figure 2, k is fixed on the boundary of $D_1 \times I$ and the region labeled h is collapsed into

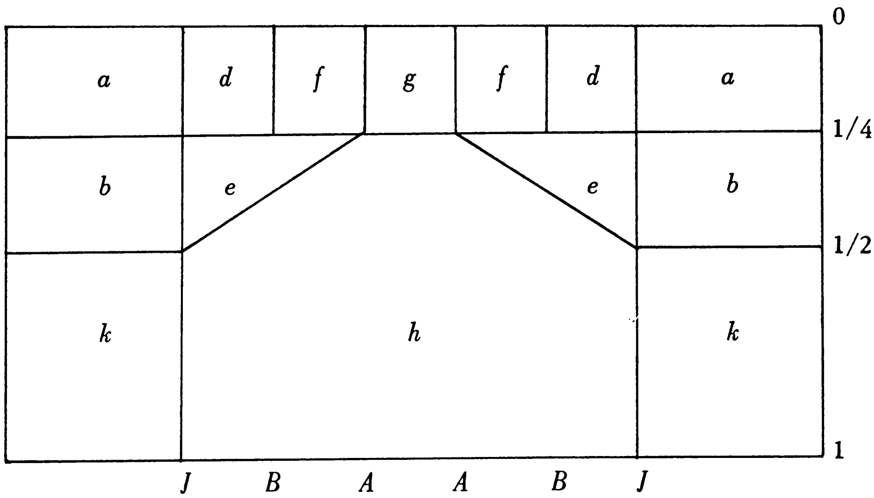

FIGURE 1

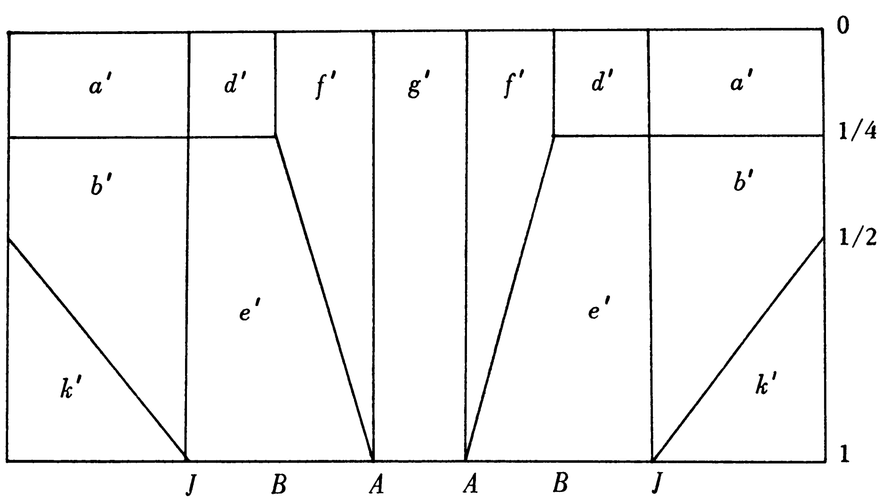

FIGURE 2

$D_1\times 1$. The map k may be extended to map $D_2\times I$ onto itself in the same manner as k maps $D_1\times I$ onto itself. The spaces $fk(\text{Int } D_i\times t)$, $t=1/2, 1/4$; $i=1, 2$, are each homeomorphic to E^2 because they are homeomorphic to spaces of cellular upper semicontinuous decompositions of E^2, by Lemma 3(2) and [14]. Let E_1 denote the annulus

$$f\{([\text{Bd } D_1, i_1(B)]\times 0)\cup(i_1(B)\times[0, 1/4])\cup k([i_1(B), i_1(A)]\times 1/4)\}\cup \text{Bd }\Delta_1$$

and T_1 the torus

$$f\{((\text{Bd } D_1)\times[0, 1/2])\cup k([\text{Bd } D_1, i_1(J)]\times 1/2)\}\cup[\text{Bd }\Delta_1, \text{Bd }\Delta_2]\cup E_1.$$

It follows from Lemmas 5.1 and 5.2 of [13] that T_1 is tame. Let E_2 and T_2 be the corresponding annulus and torus in U_2. Let M_1 be $f(M\times 0)$ minus $f(\text{Int } D_1\times 0)\cup f(\text{Int } D_2\times 0)$ plus $E_1\cup E_2$. A homeomorphism β may be defined to map $(E_1\cup E_2)\times I$ onto the tori T_1 and T_2 plus their interiors such that the extension of β on $M_1\times I$ agrees with f and yields a MCN of $C_0-\text{Int }\Delta_2$.

The MCN of $C_0-\text{Int }\Delta$. We show there exists a map P of X onto itself which collapses the annulus $[\text{Bd }\Delta_2, \text{Bd }\Delta]$ onto $\text{Bd }\Delta$, P is a homeomorphism on $X-[\text{Bd }\Delta_2, \text{Bd }\Delta)$, and P moves no point of $X-f(F^{-1}(\text{Int }\Delta)\times I)$. Letting $H_1=P\beta$ will give us that $H_1(M_1\times I)$ is a MCN of $C_0-\text{Int }\Delta$. The following spaces are described in cylindrical coordinates in E^3. Let S be the simple closed curve ($r=1/4$, $z=0$). Let L be the solid annulus ($1/4\leqq r\leqq 1$, $-1\leqq z\leqq 1$). Let P' be the map of L onto itself defined by

$$\begin{aligned} P'(r, \theta, z) &= (r+r(1-|z|), \theta, z), && 0\leqq r\leqq 1/2,\\ &= (r+(1-r)(1-|z|), \theta, z), && 1/2\leqq r\leqq 1.\end{aligned}$$

The map P' is a homeomorphism on $\text{Bd } L$ and collapses the annulus ($1/2\leqq r\leqq 1$, $z=0$) into the simple closed curve ($r=1, z=0$). Let S_1 be a simple closed curve lying in the annulus $[\text{Bd }\Delta_1, \text{Bd }\Delta_2]$ concentric to $\text{Bd }\Delta_1$. There exists a homeomorphism α of the annulus ($1/4\leqq r\leqq 1, z=0$) onto the annulus $[S_1, \text{Bd }\Delta]$ in Δ such that $\alpha(1/4, \theta, 0)\in S_1$, $\alpha(1/2, \theta, 0)\in \text{Bd }\Delta_2$ and $\alpha(1, \theta, 0)\in\text{Bd }\Delta$, for every θ. Since $\text{Int }\Delta$ is locally tame there exists a homeomorphism g of L into X such that g extends α and $(g(L)-\text{Bd }\Delta)\subset\beta(M_1\times(0, 1])\cap f(i_1F^{-1}(\text{Int }\Delta)\times I)$. The required map P is: $P(w)=gP'g^{-1}(w)$ for $w\in g(L)$ and the identity elsewhere. This completes the proof.

3. **1-sided MCN's.** Let L be a 2-manifold with boundary in a 3-manifold X. Let $x\in L$, U a 1-sided neighborhood of x and N, O_1 and O_2 given for U. We say $X-L$ is *locally simply connected on the U side* of L at x if for every $\varepsilon>0$, there exists a neighborhood $N(\varepsilon)$ of x from the local separation theorem such that $N(\varepsilon)\subset N$ and any simple closed curve in $O_1(\varepsilon)$ can be shrunk to a point in $X-L$ on a set of diameter less than ε. If $x\in\text{Int } L(x\in\text{Bd } L)$ then L is said to be *locally tame from the U side* at x if x has a neighborhood in U homeomorphic to a 3-cell

(C is locally tame at x). It follows from Theorems 4 and 8 of [5] that L is locally tame from the U side at x if L is locally simply connected on the U side at each point in a neighborhood of x. Let D be a disk. A point $x \in L$ is said to have a *1-sided MCN*, $U=f(D\times I)$, if there exists a map $f: D\times I\to X$ such that $f(D\times 1)\subset L$, $f|D\times[0, 1)$ is a homeomorphism into $X-L$, and U is a 1-sided neighborhood of x.

THEOREM 4. *Suppose L is a 2-manifold with boundary in a 3-manifold X. If $x\in L$ and x has a 1-sided MCN then L is locally tame from the U side at x.*

Proof. The proof of Lemma 3(3) essentially shows that $X-L$ is locally simply connected on the U side at x for $x\in \text{Int } L$. The case for $x\in \text{Bd } L$ is a consequence of Theorem 1. Consider a small neighborhood of x in L as the topological complex and let X be a properly chosen subset of the 1-sided MCN.

We give a short proof of a result which is part of the folklore of upper semicontinuous decompositions.

THEOREM 5. *Suppose L is a 3-manifold with boundary and G an upper semicontinuous decomposition of L all of whose nondegenerate elements lie in* Bd L *and are cellular in* Bd L. *Then L/G is a 3-manifold with boundary.*

Proof. If G is an upper semicontinuous decomposition of E^3_+ all of whose nondegenerate elements lie in E^2 and are cellular in E^2, then E^2/G has a neighborhood in E^3_+/G homeomorphic to E^3_+. For consider $E^3_+\subset E^3$; then E^2/G is homeomorphic to E^2 by [14] and E^3/G is homeomorphic to E^3 by [8]. Let P denote the projection map of E^3 onto E^3/G. For each $x\in E^2/G$, there exists a disk D such that $P^{-1}(x)\subset \text{Int } D$. Let $U=\{(x, y, z) : (x, y, 0)\in D,\ 0\leqq z\leqq 1\}$. Then $P(U)$ is a 1-sided MCN of x in E^3_+/G. By Theorem 4, x has a 3-cell neighborhood in $P(U)$. Hence E^3_+/G contains a neighborhood of E^2/G homeomorphic to E^3_+.

Let h be the projection map of L onto L/G and $x\in \text{Bd } L/G$. There exists a neighborhood Q of $h^{-1}(x)$ in Bd L which is homeomorphic to E^2 and is the union of elements of G. There exists a neighborhood B of Q in L homeomorphic to E^3_+. By the above, $h(B)$ contains a neighborhood of x in L/G homeomorphic to E^3_+. Thus L/G is a 3-manifold with boundary.

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